ON MONOTONE AND QUASICOMPACT MAPPINGS

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ABSTRACT
In this paper, some properties of monotone mappings and quasi-compact mappings have been studied.

Let $(X,\mathscr{T})$ and $(Y,\mathscr{U})$ be topological spaces. A mapping $f: X \to Y$ is said to be monotone if $f^{-1}(y)$ is connected for every point $y \in Y$. Also, f is said to be quasicompact [2] if the image under f of every inverse open subset of X is open (A subset S of X is said to be an inverse set if $f^{-1}(f(S)) = S$). Continuous quasicompact mappings are often called quotient mappings, factor mappings or identifications and these have been studied by G. T. Whyburn [6, 7], P. McDougle [2], A. H. Stone, A. V. Martin and others. Monotone mappings have been studied by G. T. Whyburn [6,7] and others. In Section 1 of this paper we have studied some properties of monotone mappings and in Section 2, some properties of quasicompact mappings. We do not assume mappings (even the quasicompact ones) to be continuous unless otherwise stated.

1. Monotone mappings

Our first result is an improvement of the well-known result that the inverse image of a connected set under an open (or closed) monotone mapping is connected.

DEFINITION. A mapping $f: X \to Y$ is said to have property P_1 if for each $y \in Y$ and each open set U such that $f^{-1}(y) \subseteq U$, y belongs to the interior of $f(U)$ [2].

P_1 mappings have been called pseudo-open by Arhangel'skii.

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Every open mapping and every closed mapping is a P_1 mapping. Also, every P_1 mapping is quasi-compact.

THEOREM 1. *Inverse images of connected sets under a P_1 monotone onto mapping are connected.*

PROOF. Consider any connected subset K of Y. Let G be a relatively closed as well as relatively open subset of f^{-1}(K). Since f is a monotone mapping, therefore 0 must be an inverse set. Also, since f is P_1, therefore the restriction of f to any inverse subset of X has property P_1 and from [2] we know that the restriction of f to $f^{-1}(K)$ is quasicompact. Consequently $f(G)$ is closed as well as open in K. Since K is connected, this is possible only if $f(G) = K$ i.e., only if $G = f^{-1}(K)$. As $f^{-1}(K)$ has no proper closed and open subset, therefore it is connected.

DEFINITION. Let $f: X \xrightarrow{\text{onto}} Y$ be any mapping. $A \subseteq X$ is said to be a trace of $B \subseteq Y$ if $f(A) = B$.

THEOREM 2. *Let $f: X \xrightarrow{\text{onto}} Y$ be a monotone mapping. A necessary and sufficient condition for the inverse images under f of connected subsets of Y to be connected subsets of X is that every connected subset of Y has a connected trace.*

PROOF. The necessity part is obvious. To prove the sufficiency of the condition, consider any connected subset K of Y. According to our assumption, there exists a connected subset H such that $f(H) = K$. Consider any point $x \in f^{-1}(K)$. Then, as f is a monotone map, $f^{-1}(f(x))$ must be a connected subset, which has nonvoid intersection with the connected set H. Therefore

$$H \cup [\, \cup \{f^{-1}(f(x)) : x \in f^{-1}(K)\}] = f^{-1}(K)$$

is a connected subset of X.

Our next result is a variant of Theorem 3.9 of W. J. Pervin and N. Levine [4] which says that a biconnected mapping from a Hausdorff space onto a semi-locally connected Hausdorff space is continuous.

DEFINITION. A mapping $f: X \to Y$ is said to be connected if the image under f of every connected subset of X is connected in Y.

THEOREM 3. *Let $(X, \mathscr{T})$ be a locally connected space and $(Y, \mathscr{U})$ be a locally connected locally compact Hausdorff space. If $f: X \xrightarrow{\text{onto}} Y$ is a monotone connected mapping with property P_1, then f must be continuous.*

PROOF. Consider any point $x \in X$. Let U be any open set containing the point $y = f(x)$. Since Y is locally compact, there exists an open set V such that $y \in V \subseteq \bar{V} \subseteq U$ and $\bar{V}$ is compact. Then $frV (= \bar{V} \cap \overline{Y \sim V})$ is a compact subset which does not contain the point y. But Y is a locally connected Hausdorff space. Therefore for each $z \in frV$ we can choose a connected open set U_z such that $z \in U_z$ and $y \in \bar{U}_z$. This family $\{U_z : z \in frV\}$ is an open cover of the compact set frV. Consequently, there exists a finite subcover $\{U_i : = 1, 2, \cdots n\}$ of frV. Also $f^{-1}(y) \subseteq X \sim \bigcup_{i=1}^{n} f^{-1}(\bar{U}_i)$. Each $f^{-1}(\bar{U}_i)$ is a connected set by virtue of Theorem 1. Let p be an adherent point of $f^{-1}(\bar{U}_i)$ for some i. Then $\{p\} \cup f^{-1}(\bar{U}_i)$ is a connected subset of X. Therefore its image under the connected mapping f must be connected in Y i.e., $f(p) \cup \bar{U}_i$ is connected in Y. But as Y is a Hausdorff space, $f(p) \cup \bar{U}_i$ can be connected only if $f(p) \in \bar{U}_i$. So $p \in f^{-1}(\bar{U}_i)$. Therefore $f^{-1}(\bar{U}_i)$ is a closed connected subset of X for all i. Consequently $X \sim \bigcup_{i=1}^{n} f^{-1}(\bar{U}_i)$ is an inverse open set containing $f^{-1}(y)$. Let C be the component of $X \sim \bigcup_{i=1}^{n} f^{-1}(\bar{U}_i)$ which contains the connected set $f^{-1}(y)$. Since X is locally connected, therefore C is open. Also, since the mapping f is monotone, C must be an inverse set. Therefore $f(C) \cap (\bigcup_{i=1}^{n} \bar{U}_i) = \emptyset$ and hence $f(C) \cap frV = \emptyset$ or $f(C) \subseteq V \cup Y \sim \bar{V}$. Recalling that $f(C)$ is connected, and $y \in f(C) \cap V$ we find that $f(C) \subseteq V \subseteq U$ or $y \in f(C) \subseteq V \subseteq U$. Hence f is continuous at the point x.

COROLLARY. *If $(X, \mathscr{T})$ is a locally connected space and $(Y, \mathscr{U})$ is a locally connected locally compact Hausdorff space then every* (1-1) *open connected mapping of X onto Y is a homeomorphism.*

The next theorem is a sharpened form of a theorem of S. Hanai [1].

DEFINITION. A topological space $(X, \mathscr{T})$ is said to be semi-compact at a point x if every neighbourhood U of x contains a neighbourhood V of x such that the frontier of V is compact. $(X, \mathscr{T})$ is said to be semicompact if it has this property at every point. Semicompact spaces are also often called rim-compact or locally peripherally compact.

THEOREM 4. *If $f: X \xrightarrow{\text{onto}} Y$ is an open monotone continuous mapping, where $(X, \mathscr{T})$ is semicompact and $(Y, \mathscr{U})$ is Hausdorff, then $(Y, \mathscr{U})$ is also a semicompact space.*

PROOF. Consider any point y of $(Y, \mathscr{U})$. Let U be any open neighbourhood of y in Y. Let x be any point belonging to $f^{-1}(y)$. Then according to the definition of a semicompact space, there must be an open set V such that $x \in V \subseteq f^{-1}(U)$ frV is compact. Since V is an open mapping, the set $W = f(V)$ is open and it is easy to see that

(1) $fr\ W \subseteq f[fr\,f^{-1}(W)]\ W$.

Again if $p \in W$ then either $f^{-1}(p)$ is contained in $\bar{V}$ or $f^{-1}(p) \cap X \sim \bar{V} \neq \emptyset$. But as f is a monotone mapping the second possibility can be true only if $f^{-1}(p) \cap frV \neq \emptyset$ i.e. if $f^{-1}(p) \subseteq f^{-1}[f(frV)]$.

Thus in both the cases we have $f^{-1}(p) \subseteq \bar{V} \cup f^{-1}\ [f(frV)]$. Hence $f^{-1}(W) \subseteq \bar{V} \cup f^{-1}(f(frV))$. Now $f(frV)$ is compact and so it is closed in Y. Therefore $f^{-1}[f(frV)]$ is closed in X. So $\bar{V} \cup f^{-1}[f(frV)]$ is a closed subset of X.

Therefore
$$\begin{aligned} frf^{-1}(W) &\subseteq [\bar{V} \cup f^{-1}(f(frV))] \sim V \\ &\subseteq frV \cup f^{-1}(f(frV)) \\ &= f^{-1}(f(frV)), \end{aligned}$$
or,
$$f[frf^{-1}(W)] \subseteq f(frV).$$

Hence from (1) we find that $frW \subseteq f(frV)$. But frV is compact and so is (frV). Also, frW being a closed subset of a compact set is compact. We already know that W is open and $y \in W \subseteq U$; consequently the space Y is semi-compact.

Our next result is again a sharpened form of a theorem of Hanai [1] This result was proved by Morita [3] with the additional assumption that $(X, \mathcal{T})$ is Hausdorff. Also, this theorem is due to Stone [5] for the case when $(X, \mathcal{T})$ is metrizable.

THEOREM 5.* *If f is a quasicompact, monotone continuous mapping from a semicompact space $(X, \mathcal{T})$ onto a Hausdorff space $(Y, \mathcal{U})$ such that $fr\,f^{-1}(y)$ is compact for each $y \in Y$, then f is a closed mapping.*

PROOF. It can be proved by using an argument similar to the proof of theorem 2 in [5] by replacing the sequence by a net and using a cluster point instead of a convergent subsequence.

2. Quasicompact mappings

It has been shown by G. T. Whyburn [6] that the restriction of a quasicompact continuous mapping $f: X \xrightarrow{\text{into}} Y$ to any inverse subset of X is quasicompact

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* In the original version, the authors had proved this theorem for P_i mappings. It was pointed out by the referee that the theorem holds for quasi compact mappings.

provided that Y is a first countable Hausdorff space. Since k-spaces include all first countable Hausdorff spaces, it is natural to see if the result holds when we take Y to be a Hausdorff k-space. (A space $(X, \mathscr{T})$ is said to be a k-space if a subset F of X is closed in X if and only if F intersects every compact subset of X in a compact subset).

A space $(X, \mathscr{T})$ is said to be a hereditary k-space if every subspace of X is a k-space.

THEOREM 6. *If f is a quasicompact and continuous mapping from a space X onto a Hausdorff hereditary k-space Y, then the restriction of f to any inverse subset of X is quasicompact.*

PROOF. Let $f^{-1}(A)$ be any inverse set in X and $f^{-1}(B)$ be an inverse set which is relatively closed in $f^{-1}(A)$. The subspace A is a k-space. If B is not relatively closed in A then there exists a compact subset C where $C \subseteq A$ but $C \cap B$ is not closed in Y. From the continuity of f, $\overline{f^{-1}(C \cap B)} \subseteq f^{-1}(\overline{C \cap B})$ (closures are taken in X and Y respectively). But $f^{-1}(\overline{C \cap B}) \subseteq f^{-1}(C) \subseteq f^{-1}(A)$. Therefore $\overline{f^{-1}(B \cap C)} \subseteq f^{-1}(C) \subseteq f^{-1}(A)$ or $\overline{f^{-1}(B \cap C)} = \overline{f^{-1}(B \cap C)} \cap f^{-1}(A)$ $\subseteq \overline{f^{-1}(B)} \cap f^{-1}(A) = f^{-1}(B)$, because $f^{-1}(B)$ is relatively closed in $f^{-1}(A)$. Combining these results, $\overline{f^{-1}(B \cap C)} \subseteq f^{-1}(B \cap C)$. Therefore $f^{-1}(B \cap C)$ is an inverse closed subset of X whose image $B \cap C$ is not closed in Y. Thus we arrive at a contradiction. Therefore image of every relatively closed inverse subset of $f^{-1}(A)$ is closed in A. Hence the result.

THEOREM 7. *If $f: X \xrightarrow{\text{onto}} Y$ is a quasicompact monotone continuous mapping and Y is a Hausdorff hereditary k-space, then the inverse of every connected subset of Y is connected.*

PROOF. Let K be any connected subset of Y. Suppose $f^{-1}(K)$ is disconnected. Then $f^{-1}(K) = C \cup D$ where C and D are disjoint non-empty sets relatively open in $f^{-1}(K)$. Since f is a monotone mappng, the sets C and D must be inverse sets. Also from Theorem 6, the restriction of f to $f^{-1}(K)$ is quasicompact. Therefore $f(C)$ and $f(D)$ are disjoint relatively open sets and $K = f(C) \cup f(D)$. This is not possible because K is connected. Hence the result.

THEOREM 8. *If $f: X \xrightarrow{\text{onto}} Y$ is a quasicompact continuous mapping where $(X, \mathscr{T})$ is a locally compact locally connected Hausdorff space and $(Y, \mathscr{U})$ is any Hausdorff space, then f is a closed mapping provided that $f^{-1}(y)$ is a* *compact connected set for each $y \in Y$. Consequently $(Y, \mathscr{U})$ is a locally connected locally compact space.*

PROOF. Follows easily in view of Theorem 5 of Section 1.

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